Different rank theorems

DEF: Regular submanifold of dim k. $S \subseteq N$ of dim n is a regular submanifold of dim k, if for every $p \in S$, $\exists$ a coordinate neighborhood $(U, \phi) = (U, x^1, x^2, \cdots x^n)$ s.t. $U \cap S$ is defed by the vanishing of $n-k$ of the coordinate functions. WLOG, we assume $x^{k+1} = x^{k+2} = \cdots = x^n = 0$. $n-k$ is the codimension of S in N.

Thm: $S \subseteq N$ is a regular submanifold. $\dim S = k$. (if vanishing $n-k$ coordinates)

DEF: Regular level set: c is regular $\Leftrightarrow$ $\begin{cases} c \text{ is not the image} \\ \text{for } F^{-1}(c)\text{'s point } p, \ F_{*p} \text{ is surjective.} \end{cases}$

The inverse image of a regular value c is called a regular level set

if 0 is regular value. $F^{-1}(0)$: regular zero set

Thm: $g: N \longrightarrow \mathbb{R}$ is a C^∞ function on manifold N. Then a nonempty regular level set $S = g^{-1}(c)$ is a regular submanifold of N of codimension 1.

Thm (Regular Level Set Thm) $F: N \longrightarrow M$ is a C^∞ map of manifolds. $\dim N = n$. $\dim M = m$. A nonempty regular level set $F^{-1}(c)$, where $c \in M$, is a regular submanifold of N of dim equal to $n-m$.

(Examples): $GL_n(\mathbb{R}) \supseteq SL_n(\mathbb{R})$ is a regular submanifold of codim 1.

$\Rightarrow \dim SL_n(\mathbb{R}) = \dim GL_n(\mathbb{R}) - 1 = n^2 - 1$

proof: $f: GL_n(\mathbb{R}) \longrightarrow \mathbb{R}$, $SL_n(\mathbb{R}) = f^{-1}(1)$

$[a_{ij}] = A \longmapsto |A|$

denote $m_{ij} = \det S_{ij}$. the (i,j)-minor of A

$\frac{\partial f}{\partial a_{ij}} = (-1)^{i+j} m_{ij}$, $A \in GL_n(\mathbb{R})$ is critical $\Leftrightarrow m_{ij} = 0$, $\begin{matrix} 1 \le i \le n \\ 1 \le j \le n \end{matrix}$

$\Rightarrow$ all matrices in $SL_n(\mathbb{R})$ are regular points.

DEF: (immersion) $f: N^n \longrightarrow M^m$. $n \le m$, f has rank n at p (f injective)

(submersion) $f: N^n \longrightarrow M^m$ $n \ge m$. f has rank m at p.

Thm (Constant Rank Thm) $f: N^n \longrightarrow M^m$ has constant rank k in a neighborhood of a point p. Then $\exists$ $\begin{cases} (U, \phi) \text{ local chart centered at } p \\ (V, \psi) \text{ local chart centered at } f(p) \end{cases}$

s.t. $(\psi\circ f\circ\phi^{-1})(r^1, r^2\cdots r^n) = (r^1, r^2, \cdots r^k, 0\cdots 0)$, where $(r^1\cdots r^n)\in\phi(U)$.

Thm: (constant-rank level set thm) $f: N^n \longrightarrow M^m$ is a C^∞ map of manifolds. $c\in M$.
f has constant rank k in a neighborhood of the level set $f^{-1}(c)$ in N.
Then $f^{-1}(c)$ is a regular submanifold of N of codim k.
记忆：无妨最简单 $f: (x^1, x^2\cdots x^n) \longmapsto (x^1, x^2, \cdots, x^k, 0\cdots 0)$

$$f^{-1}(c) = (\underbrace{x^1, x^2, \cdots x^k}_{\text{fixed}}, \underbrace{\mathbb{R}, \mathbb{R}, \cdots \mathbb{R}}_{n-k \text{ copies of } \mathbb{R}})$$

Example: $O(n)$ is a regular submanifold of $GL_n(\mathbb{R})$

$$f: GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R}) \qquad O(n) = f^{-1}(I).$$
$$A \longmapsto A^T A.$$

denote left multiplication / Right multiplication as l_c, r_c.

$(f\circ r_c)(A) = (l_{c^T}\circ r_c\circ f)(A)$

$\Rightarrow f_{*,AC}\circ \underline{(r_c)_{*,A}} = \underline{(l_{c^T})_{*,A^TAC}}\circ \underline{(r_c)_{*,A^TA}}\circ f_{*,A}$

$\Rightarrow r_c, l_c$ are diffeomorphism $\Rightarrow$ differentials are isomorphisms

$\Rightarrow \operatorname{rk} f_{*,AC} = \operatorname{rk} f_{*,A}$ $\forall A$ and C

$\Rightarrow$ f has constant rank on $GL_n(\mathbb{R})$

$\Rightarrow f^{-1}(I)$ is a regular submanifold of $GL_n(\mathbb{R})$

Next: immersions and submersions have constant rank.

Thm: $f: N^n \longrightarrow M^m$ immersion: it has constant rank n
submersion: it has constant rank m.

Details:

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "\phi"'] & M \arrow[d, "\psi"] \\ U \arrow[r] & V \end{tikzcd}$$

immersion: $(\psi\circ f\circ\phi^{-1})(r^1\cdots r^n) = (r^1, \cdots r^n, 0\cdots 0)$

submersion: $(\psi\circ f\circ\phi^{-1})(r^1, \cdots r^m, \cdots r^n) = (r^1\cdots r^m)$

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